# Theoretical Modeling of an Ideal Photon Gas

1. V Nolting

†Department of Applied Physical Sciences, Vaal University of Technology, Vanderbijlpark, South Africa

†E-mail: noltiv@gmail.com

.

**Abstract.** In this paper the correct quantum statistical description of an ideal photon gas is presented. It is shown that the ideal photon gas undergoes Bose-Einstein condensation for both three and two dimensional systems with a finite critical temperature $T_C \neq 0$. However, there is no phase transition observed in the case of a one-dimensional system. The phase transition is modeled using a quantum statistical analogy to the Clausius Clapeyron equation. Furthermore, from the heat released during condensation the heat capacity $C_V$ is calculated as a function of temperature $T$. The famous $\lambda$-shaped peak at $T = T_C$ is reproduced which identifies the phase transition as one of 1st order. At high temperatures the results of classical thermodynamics are reproduced.

## 2. Introduction

Time dependent electric and magnetic fields give rise to electromagnetic waves. The energy released in electromagnetic radiation is quantized; the corresponding fundamental particles are photons. Photons are bosons and are described by the Bose Einstein statistics. One important feature of the boson quantum gas is Bose-Einstein condensation when a finite macroscopic fraction of bosons occupy the ground state. The condensation of bosons into the ground state is described by the equation

$$\frac{n_0}{n} = 1 - \left(\frac{T}{T_C}\right)^{3/2}$$

The corresponding phase transition was first theoretically predicted by A Einstein [1] and then experimentally confirmed by E A Cornell et al [2] as a phase transition of 2nd order.

Bose-Einstein condensation of photons has been a controversial subject because (1) photons are massless particles and (2) the particle number of photons is generally not conserved. However, these difficulties can be overcome if one considers a photon gas in a Fabry-Perot cavity filled with a gaseous medium in order to achieve thermodynamic equilibrium. In this way the photon mass becomes effective due to the system now being finite. Additionally, absorption and emission processes of photons can be balanced so that the total number of

photons becomes conserved. Along these lines of thought Bose-Einstein condensation of light has been experimentally observed [3,4] and the presence and properties of the medium inside the cavity is important in this respect. From a statistical description of the equilibrium state the authors of reference [5] evaluate the critical temperature $\boldsymbol{T_C}$ of the photon gas as a function of the number of photons in the system. It is claimed in ref [6] that an ideal photon gas undergoes Bose-Einstein condensation for dimensions $\boldsymbol{d = 2}$ and $\boldsymbol{d = 3}$ only. However, it is questionable whether these conclusions are generally applicable. Additionally, a system of non-interacting particles can in reality not be realized as even the slightest photon-photon interactions may produce enough heat for the system to transition into the gaseous phase.
A lot of research in recent years is dedicated to the analogy between LASER physics and Bose Einstein condensation of quasiparticle systems like photons [7,8]. Bose Einstein condensation in LASER systems is an interesting topic due to their potential technological applications. In this article we want to contribute to a better understanding of this interesting field of study by investigating the phase transition in more detail. We present the correct quantum statistical description of the photon gas and the dependence of Bose Einstein condensation on the dimension $d$ of the system. The article is organized as follows. In Section 2 the theory of the photon gas is presented and the critical temperature $T_C$ is calculated as a function of particle density $n$. It is shown that the phase transition into the condensed phase depends on the dimension $d$ of the system; analogies to the theory of magnetism are discussed. Conclusions are drawn in Section 4.

## 3. Theory

In this section the theory of the photon gas is developed. The radiation field is described by the homogeneous wave equation

$$\left(\Delta - \frac{1}{c^2}\ \frac{\partial^2}{\partial\, t^2}\right)\ \boldsymbol{\Psi}\,(\boldsymbol{r}, \boldsymbol{t}) = \boldsymbol{0} \tag{1}$$

Here $\Psi\ (\boldsymbol{r}, t)$ denotes the space and time dependent wave function while $c$ is the speed of light in vacuum. The solution to Eq (1) is decomposed with respect to plane waves so that

$$\Psi\ (\boldsymbol{r}, t) \rightarrow\ \Psi\ (\boldsymbol{k}, t)\ \ e^{i\ \boldsymbol{k}\cdot\boldsymbol{r}} \tag{2}$$

Inserting Eq (2) into Eq (1) yields the equation of motion

$$\left(\frac{\partial^2}{\partial\, t^2} + (\ \boldsymbol{k}^2\ \ c^2\ )\right)\ \ \Psi\ (\boldsymbol{k}, t) = 0 \tag{3}$$

of a linear harmonic oscillator with the $k$-dependent frequencies $\omega\ (\boldsymbol{k}) = c\ |\boldsymbol{k}|$ and the discrete energy eigenvalues

$$E_n\ =\ \hbar\, c\, |\boldsymbol{k}|\ \left(n + \frac{1}{2}\right) \tag{4}$$

$n = 0,1,2,3, \dots \dots$. Each photon then has the energy

$$E\ =\ \hbar\, c\ \ |\boldsymbol{k}|\ = cp \tag{5}$$

and the rest mass $m_0\ = 0$.
The average number of photons is calculated from the parabolic density of states $\rho\ (E) \sim\ E^2$ if $E\ \geq 0$. We then obtain

$$\langle N \rangle\ =\ \int_{-\infty}^{+\infty} \rho\ (E)\ \ f_+\ (E, T)\ dE\ \ \ \cong\ \ 2.032\ \cdot\ \ 10^7\ \ V\ \ T^3 \tag{6}$$

Here $f_+\ (E, T) = \frac{1}{e^{\beta\,(E-\,\mu\,)}+1}$ denotes the Bose-Einstein distribution function. It follows from Eq (6) that the average number of photons is temperature dependent. In particular

$$\langle N \rangle\ (T = 0)\ \ = 0 \tag{7}$$

On the other hand, the internal energy of the photon gas is calculated from

$$U(T, V) =\ \int_{-\infty}^{+\infty} E\ \ \rho\ (E)\, f_+\ (E, T)\ dE\ \ =\ \sigma\, V\, T^4 \tag{8}$$

Eq (8) is known as Stefan-Boltzmann's law; $\sigma$ is the Stefan-Boltzmann constant. Alternatively to Eq (8) the energy density $U/_V$ of the photon gas can also be expressed via the spectral energy density $\varepsilon\ (\omega, T)$ as an integral over all frequencies $\omega$.

$$U/_V = \int_0^\infty \varepsilon\ (\omega, T)\ d\omega \tag{9}$$

The spectral energy density is given by Planck's law.

Let's consider Bose Einstein condensation of an ideal photon gas. The particle density $n_0$ of the ground state $\varepsilon\ (k = 0) = 0$ can be written as

$$n_0 = n - \frac{2S+1}{\lambda^3\ (T)}\ g_3\ (z) \tag{10}$$

Here $\lambda\ (T)$ denotes the thermal de Broglie wavelength. The transition into the condensed phase is regulated by the condition

$$n\ \lambda^3\ = (2S+1)\ g_3\ (z = 1) \tag{11}$$

and critical data are obtained from $n_0/_n\ = 0$. It then follows

$$n = \frac{2S+1}{\pi^2\ (\beta_c\ \hbar\ c)^3}\ g_3\ \ (z = 1)$$

$$k_B\ T_C\ (n) = \hbar\ c\ \left(\frac{\pi^2\ n}{(2S+1)\ g_3\ (z=1)}\right)^{1/3} \tag{12}$$

Here the polynomial $g_\alpha\ (z)$ is defined as the infinite series

$$g_\alpha\ (z) = \sum_{n=1}^\infty \frac{z^n}{n^\alpha} \tag{13}$$

As $g_3\ (z = 1)$ defines the Riemann ξ-function via

$$g_3\ (z = 1) = \sum_n \frac{1}{n^3}\ =\ \xi\ (3)\ \ \cong 1.2021 \tag{14}$$

it follows that there is a finite critical temperature $T_C\ (n) \neq 0$ in the case of the three dimensional photon gas. A phase transition into the condensed phase is observed for this particular case. Furthermore, evaluating Eq (12) further yields for the critical temperature of the photon gas the particle density dependent result

$$T_C\ (n) = 0.3197\ \ n^{1/3} \tag{15}$$

if $n$ is measured in $cm^{-3}$. The increase of $T_c$ with $n$ qualitatively agrees with the results of A Kruchkov [5]. In this way critical temperatures close to absolute zero can be obtained. However, it turns out that the phase transition depends on the dimension $d$ of the photon gas.

For $d = 2$ and $d = 1$ the corresponding infinite series in the denominator of Eq (12) are

$$g_2\ (z = 1) = \sum_n \frac{1}{n^2}\ =\ \xi\ (2) = \ \frac{\pi}{6}$$

$$g_1\ (z = 1) = \sum_n \frac{1}{n}\ \ \rightarrow\ \infty \tag{16}$$

In the one dimensional system the series diverges so that $T_C = 0$. While the two dimensional system shows Bose Einstein condensation with a finite critical temperature $T_C\ \neq 0$ there is no such phase transition in the case of a one dimensional system. Our results confirm the findings of reference [5,9].

The one dimensional system is an interesting test case as it shows how photons behave on a quantum level. Similar dependencies of a phase transition on the dimension $d$ of the system are also observed in the case of magnetic systems (see Mermin-Wagner theorem, Heisenberg and Ising model [10, 11] ).. Generally according to reference [12] the existence of a phase transition depends on the dimension $d$ of the system and the range of particle interactions. Even though strong particle interactions are more likely to cause phase transitions, phase transitions in non-interacting particle systems have been observed if particle movement is restricted and the particles bunch and stick together while entering the same energy state.

In the next section we will discuss how these phase transitions affect the behavior of the heat capacity $C_V\ (T)$ of the photon gas.

## 3. Results

3.1 The entropy function

The entropy function $S\ (T)$ is calculated from the grand canonical potential and corresponding results are plotted in Fig 1 below.

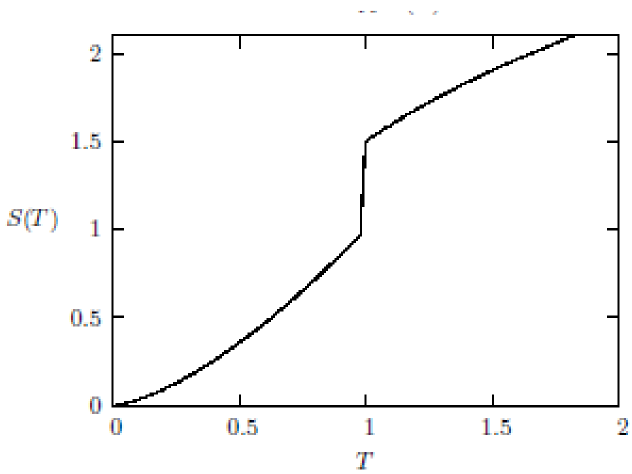


**Figure 1.** Entropy $S\ (T)$ as a function of reduced temperature $T/T_c$.

The entropy function $S\ (T)$ is discontinuous at $T = T_c$ with
$S\ (T) \sim T^{3/2}$ for $T < T_c$ which is the condensed phase and
$S\ (T) \sim \ln T$ for $T > T_c$ which is the vapor phase.
From the entropy difference the heat released per particle during condensation can be computed. Alternatively $\Delta Q$ can also be expressed using the Clausius Clapeyron equation and the slope of the vapor pressure curve. For the $3d$-system we then obtain

$$\Delta Q = \frac{5}{2}\ k_B\, T\ \frac{g_{5/2}\,(z=1)}{g_{3/2}\,(z=1)} \quad (14)$$

Using similar conclusions as in the previous section it follows that for the $3d$-system $\Delta Q \neq 0$ which characterizes the phase transition as a 1st order phase transition. On the other hand, for the $2d$-systems $\Delta Q = 0$ , the entropy function $S(T)$ becomes continuous, and the phase transitions are now of 2nd order [13].

3.2 The heat capacity

The heat capacity $C_V\ (T) \sim T^{3/2}$ in the condensed phase while at the critical temperature $T_c$ a peak value of

ISSN: 0017-2294

$$C_V\ (T = T_c) \cong 1.9267\ N\ k_B$$

is reached. For $T \rightarrow \infty$ the heat capacity asymptotically reaches the classical result

$$C_V = \frac{3}{2}\ N\ k_B$$

These results are confirmed in Fig 2 below.

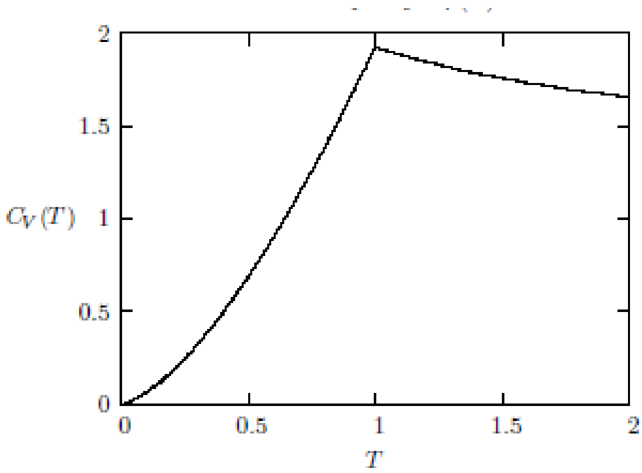


**Figure 2.** Heat capacity $C_V\ (T)$ as a function of reduced temperature $^T/_{T_c}$.

Note the $\lambda$-shaped peak at $T = T_c$ which is a result of Bose Einstein condensation. The phase transition is again one of 1st order while 2nd order phase transitions are characterized by singularities and discontinuities in the response function at the critical temperature. Similar results for $C_V\ (T)$ are also reported in reference [14]. The results of Figure 2 can be explained using the two phase theory. For $T < T_C$ two phases coexist in equilibrium, namely the condensed phase of $N_0$ particles in the ground state and the remaining $N - N_0$ particles in the gaseous phase.; for $T > T_C$ only the gaseous phase exists. The Lee Yang theory of Bose Einstein condensation [15] can be used to estimate the critical temperature $T_C$ for both $d = 2$ and $d = 3$. The theory also proves that there is no phase transition for $d = 1$ thereby confirming our results.

Comparisons with experimental values turn out to be quite a challenging task as a system of interaction free particles can in reality not be produced. However, there is a qualitative agreement with reference [16] where a cusp singularity of $C_V$ at $T = T_C$ is reported for a two-dimensional photon gas. This confirms the earlier quoted result that the two-dimensional system indeed shows a 2nd order phase transition. However, a level of uncertainty remains in the literature regarding the one-dimensional case. The authors of reference [17] report a crossover behavior from two dimensions into the one-dimensional quantum gas of light. However, in our calculations no phase transition is observed in the one-dimensional case.

## 4.Conclusions

In this paper we presented a theoretical quantum statistical description of an ideal photon gas. The phase transition from the vapor phase to the condensed phase depends on the dimension $d$ of the system. This is confirmed from calculations of the critical temperature $T_C$, the entropy function $S\ (T)$, and the heat capacity $C_V\ (T)$. For $3d$-systems this phase transition is of $1^{st}$ order which is confirmed by the $\lambda$-shaped peak in the heat capacity. On the other hand, the $2d$-system shows a second order phase transition in agreement with other authors. In the one-dimensional case no phase transition is observed.

## References

[1] A Einstein, Sitzungsbericht der Akademie der Wissenschaften 261 (1924)
[2] E A Cornell and C E Wieman, Rev Mod Phys. 74, 875 (2002)
[3] J Klaers et al, Nature Physics 6 (2010)
[4] J Klaers, Nature 468 (2010)
[5] A Kruchkov, Phys Rev A89 (2014)
[6] E E Muller, Note on Bose Einstein condensation of photons, arxiv.org/pdf/ (2018)
[7] M Pieczarka et al, Nature Photonics 18 (2024)
[8] S Barland et al, Optics Express 29 (2021)
[9] E E Muller, Annals of Physics 184 (1988)
[10] N D Mermin and H Wagner, *Phys Rev Letters* ***17,*** *1133 (1966)*
[11] W Nolting and A Ramakanth,, *Quantum Theory of Magnetism, Springer (2009)*
[12] R R Griffiths, *Phys Rev Letters 24, 1949 (1970)*
[13] P C Hohenberg, *Phys Rev 158 , 383 (1967)*
[14] W Nolting, *Theoretical Physics, Vol 8: Statistical Physics,* Springer (2018)
[15] F Brange et al, Phys Rev A107 (2023)
[16] T Damm et al , *Nature Communications. (2016)*
[17] K K Umesh et al, Nature Physics 20 (2011)